§. Haar Series Continued:

Recall the averaging formula: $\langle f\rangle_I = \sum_{J\supsetneq I}(f,h_J)h_J(I)$.
We are in $L^2(\mathbb{R})$, where we know $\{h_I\}_{I\in\mathcal{D}}$ is an orthonormal basis,

$$f=\sum_{I\in\mathcal{D}}(f,h_I)h_I$$

holds in the sense of L^2-convergence, and $\|f\|_{L^2}^2=\sum_{I\in\mathcal{D}}(f,h_I)^2$.
For every $I\in\mathcal{D}$, $f\mapsto\langle f\rangle_I$ is a continuous linear functional on $L^2(\mathbb{R})$,
so

$$\langle f\rangle_I=\Big\langle\sum_{J\in\mathcal{D}}(f,h_J)h_J\Big\rangle_I=\sum_{J\in\mathcal{D}}(f,h_J)\langle h_J\rangle_I=\sum_{J\supsetneq I}(f,h_J)h_J(I).$$

Now suppose $I\subsetneq J$ are dyadic intervals:

$$\langle f\rangle_I-\langle f\rangle_J=\underbrace{\sum_{K\supsetneq I}(f,h_K)h_K(I)}-\sum_{K\supsetneq J}(f,h_K)h_K(J)$$

Split into: $\sum_{\substack{K\supsetneq I\\ K\subset J}}(f,h_K)h_K(I)+\sum_{K\supsetneq J}(f,h_K)\underbrace{h_K(I)}_{=h_K(J),\ \forall K\supsetneq J\supsetneq I}$

$$\Rightarrow\ \langle f\rangle_I-\langle f\rangle_J=\sum_{\substack{K\supsetneq I\\ K\subset J}}(f,h_K)h_K(I)+\cancel{\sum_{K\supsetneq J}(f,h_K)h_K(J)}-\cancel{\sum_{K\supsetneq J}(f,h_K)h_K(J)}$$

$$\Rightarrow\ \boxed{\langle f\rangle_I-\langle f\rangle_J=\sum_{\substack{K\supsetneq I\\ K\subset J}}(f,h_K)h_K(I)}\qquad \forall I\subsetneq J\in\mathcal{D}\qquad (*)$$

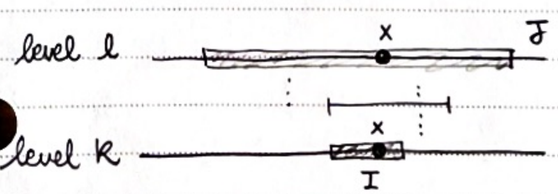
Consider now the lengths of these intervals: say $|I|=2^k$, $|J|=2^\ell$ $(k<\ell\in\mathbb{Z})$
We can sum $(*)$ over all dyadic intervals of length 2^k and 2^ℓ:

$$\sum_{\substack{I\in\mathcal{D}\\ |I|=2^k}}\langle f\rangle_I-\sum_{\substack{J\in\mathcal{D}\\ |J|=2^\ell}}\langle f\rangle_J=\sum_{\substack{I\subsetneq J\\ |I|=2^k,\,|J|=2^\ell}}\ \sum_{\substack{K\supsetneq I\\ K\subset J}}(f,h_K)h_K(I).$$

This can actually be expressed by the following useful formula:

$$\boxed{\sum_{\substack{I\in\mathcal{D}\\ |I|=2^k}}\langle f\rangle_I\mathbb{1}_I(x)-\sum_{\substack{J\in\mathcal{D}\\ |J|=2^\ell}}\langle f\rangle_J\mathbb{1}_J(x)=\sum_{\substack{K\in\mathcal{D}\\ 2^k<|K|\le 2^\ell}}(f,h_K)h_K(x).}\qquad (**)$$

Pf: Let $x\in\mathbb{R}$. Then there is a unique $I\in\mathcal{D}$ s.t. $|I|=2^k$ and $x\in I$. Similarly, there is a unique $J\in\mathcal{D}$ s.t. $|J|=2^\ell$ and $x\in J$. So the left hand side of $(**)$ becomes exactly $\langle f\rangle_I-\langle f\rangle_J$. For the right hand side, for every i s.t. $k<i\le\ell$ the unique $K\in\mathcal{D}$ s.t. $|K|=2^i$ and $x\in K$ also satisfies $K\supsetneq I$ and $K\subset J$ (b/c K intersects I & J in x, K must contain or be contained in I & J). Moreover $h_K(x)=h_K(I)$. So $(**)$ becomes exactly



$$\langle f\rangle_I-\langle f\rangle_J=\sum_{\substack{K\supsetneq I\\ K\subset J}}(f,h_K)h_K(I).$$

∎

So, again, for $x \in \mathbb{R}$:

$$\sum_{\substack{I \in \mathcal{D} \\ |I| = 2^k}} \langle f \rangle_I \mathbb{1}_I(x) - \sum_{\substack{J \in \mathcal{D} \\ |J| = 2^\ell}} \langle f \rangle_J \mathbb{1}_J(x) = \sum_{\substack{K \in \mathcal{D} \\ 2^k < |K| \leq 2^\ell}} (f, h_K) h_K(x)$$

and at most one term is non-zero in each sum on the left, and at most $(\ell - k)$ terms are non-zero on the right! So take $k \to -\infty$ and $\ell \to \infty$:

$$\boxed{\lim_{k \to -\infty} \left(\sum_{\substack{I \in \mathcal{D} \\ |I| = 2^k}} \langle f \rangle_I \mathbb{1}_I(x) \right)} - \boxed{\lim_{\ell \to \infty} \left(\sum_{\substack{J \in \mathcal{D} \\ |J| = 2^\ell}} \langle f \rangle_J \mathbb{1}_J(x) \right)} = \boxed{\sum_{K \in \mathcal{D}} (f, h_K) h_K(x)}$$

Haar Series

$\downarrow$ $f(x)$ for a.a. $x \in \mathbb{R}$ (LDT)

As $k \to -\infty$, I collapses about x and this is really

$$\lim_{\substack{I \ni x \\ |I| \to 0}} \langle f \rangle_I = f(x) \text{ for a.a. } x$$

$\downarrow$ 0 for all x

$$|\langle f \rangle_J| \leq \langle |f| \rangle_J = \frac{1}{|J|} \int_J |f|$$

$$\leq \frac{1}{|J|} \|f\|_{L^2(\mathbb{R})} \sqrt{|J|}$$

$$= \frac{\|f\|_{L^2(\mathbb{R})}}{\sqrt{|J|}} \xrightarrow[|J| \to \infty]{} 0$$

$\downarrow$ in L^2: f

So: the sum on the left converges pointwise a.e. to f $\Rightarrow$ so does the Haar series!

Prop: For all $f \in L^2(\mathbb{R})$, the Haar series $\sum_{I \in \mathcal{D}} (f, h_I) h_I$ converges to f both in L^2 and pointwise a.e.

Def.: Let $\mathcal{X}$ be a (real or complex) Banach space. A sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{X}$ is called a **Schauder basis** ("basis") of $\mathcal{X}$ if the closed linear span of $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{X}$ and $\sum_{n=1}^{\infty} a_n x_n = 0$ if and only if each $a_n = 0$.

We already saw that $\overline{\text{span}\{h_I\}_{I\in\mathcal{D}}} = L^p(\mathbb{R})$, $\forall 1<p<\infty$, and further

$$\sum_{I\in\mathcal{D}} (f,h_I)h_I = 0 \Rightarrow (h_J, \sum_{I\in\mathcal{D}} (f,h_I)h_I) = 0, \forall J\in\mathcal{D} \Rightarrow (f,h_J)=0, \forall J\in\mathcal{D}.$$

$\Rightarrow$ $\{h_I\}_{I\in\mathcal{D}}$ is a Schauder basis for $L^p(\mathbb{R})$, $1<p<\infty$.

Problem: The sum $\sum_{n=1}^{\infty} a_n x_n$ may depend on the order of x_n, and it is possible for a permutation of a basis to fail to be a basis!
Bases which remain bases under all permutations are called **unconditional bases.**

Thm.: Given a basis $\{x_n\}_{n\in\mathbb{N}}$ of a Banach space, TFAE:

(i). $\{x_{\pi(n)}\}_{n\in\mathbb{N}}$ is also a basis for all permutations $\pi: \mathbb{N}\to\mathbb{N}$.

(ii). Sums of the form $\sum_{n=1}^{\infty} a_n x_n$ converge unconditionally whenever they converge.

(iii). There is $C>0$ s.t. for every sequence of scalars $\{a_n\}_{n\in\mathbb{N}}$ and every sequence of scalars $\{\sigma_n\}_{n\in\mathbb{N}}$ with $|\sigma_n|\leq 1, \forall n$:

$$\left\| \sum_{n=1}^{\infty} \sigma_n a_n x_n \right\|_{\mathcal{X}} \leq C \left\| \sum_{n=1}^{\infty} a_n x_n \right\|_{\mathcal{X}}$$

See: [Christopher Heil - "A Basis Theory Primer" - Birkhäuser]

Our interest in this: **The Haar basis is an unconditional basis for $L^p(\mathbb{R})$, $1<p<\infty$.**
Informally: → we can approximate a function in the L^p norm w/ an infinite linear combination of Haar functions (the basis part);
→ the order of summation doesn't matter (the unconditional part).
$\Rightarrow$ L^p information about $f\in L^p(\mathbb{R})$ is encoded in $|(f,h_I)|$, with $I\in\mathcal{D}$.
$\Rightarrow$ motivates the definition:

Def.: The **martingale transform** operator T_σ is defined by:

$$\boxed{T_\sigma f := \sum_{I\in\mathcal{D}} \sigma_I (f,h_I) h_I}$$

where $\sigma = \{\sigma_I\}_{I\in\mathcal{D}}$ is a sequence of real numbers with $\|\sigma\|_\infty < \infty$.
Remark: In some texts, the martingale transform is defined with the sequence σ taking values only ± 1: $\sigma_I = \pm 1, \forall I$.

Remark: An even more general definition is that of Haar multipliers i.e. operator of the form

$$Tf(x) := \sum_{I \in \mathcal{D}} \phi_I(x) (f, h_I) h_I(x)$$

where the "symbol" $\phi_I(x)$ is a sequence of functions indexed by I.
So the martingale transform is a "constant Haar multiplier", i.e. $\phi_I(x) \equiv \sigma_I$, $\forall I$.

- Take $\phi_I(x) = g(x)$, $\forall I \Rightarrow$ multiplication by g
- Haar multipliers are analogous to pseudodifferential operators

$$\boxed{\Phi f(x) := \int_{\mathbb{R}} a(x,\xi) \hat{f}(\xi) e^{2\pi i x \xi} d\xi}$$

where the Haar system is replaced by trigonometric functions. (the "symbol" here is $a(x,\xi)$). In both situations though: one wants to identify those symbols for which the corresponding operators are bounded on L^p.

Theorem: Let $\mathcal{D}$ be a dyadic grid on $\mathbb{R}$, $\{h_I\}_{I\in\mathcal{D}}$ be the Haar system adapted to $\mathcal{D}$, and:

$$\boxed{T_\sigma f := \sum_{I\in\mathcal{D}} \sigma_I (f, h_I) h_I}$$

be a martingale transform, i.e. $\{\sigma_I\}_{I\in\mathcal{D}}$ is a sequence of real numbers indexed by $\mathcal{D}$, satisfying $|\sigma_I| \leq 1$, $\forall I\in\mathcal{D}$. Then $T_\sigma : L^p \to L^p$ is a bounded operator on $L^p(\mathbb{R})$ for all $1<p<\infty$: there is a constant C_p depending only on p s.t.

$$\boxed{\|T_\sigma f\|_{L^p(\mathbb{R})} \leq C_p \|f\|_{L^p(\mathbb{R})}.} \quad (*)$$

Remarks:

- In the language of functional analysis, this says exactly that the Haar system is an unconditional basis for $L^p(\mathbb{R})$.
- Assume for a moment that T_σ is given by a ± 1 sequence, i.e. $\sigma_I = \pm 1$, $\forall I \in \mathcal{D}$. Then $T_\sigma T_\sigma f = f$, so $\|f\|_{L^p(\mathbb{R})} = \|T_\sigma(T_\sigma f)\| \leq C_p \|T_\sigma f\|_{L^p}$, and we have an equivalence of norms $\|f\|_{L^p} \simeq \|T_\sigma f\|_{L^p}$.
- Look at what happens when $p=2$:

$$\|T_\sigma f\|_{L^2}^2 = \sum_I (T_\sigma f, h_I)^2 = \sum_I |\sigma_I|^2 (f, h_I)^2 \leq \sum_I (f, h_I)^2 = \|f\|_{L^2}$$

so $(*)$ holds with $C_2 = 1$.

We introduce a closely related dyadic operator:

Def.: The dyadic square function operator $S_\mathcal{D}$ (adapted to some dyadic grid $\mathcal{D}$ on $\mathbb{R}$) is defined by:

$$\boxed{S_\mathcal{D} f(x) := \left(\sum_{I\in\mathcal{D}} (f, h_I)^2 \frac{\mathbb{1}_I(x)}{|I|} \right)^{1/2}}$$

First immediate observation: $\|S_\mathcal{D} f\|_{L^2} = \|f\|_{L^2}$.

$$\|S_\mathcal{D} f\|_{L^2}^2 = \int \sum_{I\in\mathcal{D}} (f, h_I)^2 \frac{\mathbb{1}_I}{|I|} = \sum_{I\in\mathcal{D}} (f, h_I)^2 = \|f\|_{L^2}^2$$
(Tonelli)

Remark: $S_\mathcal{D}$ is **not** a linear operator, but it is **quasilinear**:

- $|S_\mathcal{D}(\lambda f)| = |\lambda| |S_\mathcal{D} f|$
- $S_\mathcal{D}(f+g) \leq \sqrt{2}(S_\mathcal{D} f + S_\mathcal{D} g)$

$$S_\mathcal{D}(f+g) = \left(\sum_I ((f,h_I)+(g,h_I))^2 \frac{\mathbb{1}_I}{|I|}\right)^{1/2} \leq \left(2\sum_I \left((f,h_I)^2 + (g,h_I)^2\right)\frac{\mathbb{1}_I}{|I|}\right)^{1/2}$$

$$= \sqrt{2}\left(\sum_I (f,h_I)^2 \frac{\mathbb{1}_I}{|I|} + \sum_I (g,h_I)^2 \frac{\mathbb{1}_I}{|I|}\right)^{1/2} \leq \sqrt{2}(S_\mathcal{D} f + S_\mathcal{D} g)$$

$(\sqrt{a+b} \leq \sqrt{a} + \sqrt{b})$

"if this doesn't seem remarkable, that's only because we've been spoiled by a too-close familiarity w/ functional analysis"

A closer look at $\boxed{\|f\|_2 = \|S_D f\|_2}$ (from Wilson):

→ $\boxed{\|f\|_2 \le \|S_D f\|_2}$ (1)

$$|f|^2 = \Big|\sum_I (f,h_I)h_I\Big|^2$$

$$(Sf)^2 = \sum_I (f,h_I)^2 (h_I)^2$$

would expect the square of a sum to be much bigger than the sum of squares, but (1) shows that, in the average, this isn't true for a sum of Haar functions.

Why? The sum $\sum_I (f,h_I)h_I$ has a lot of cancellation

→ $\boxed{\|S_D f\|_2 \le \|f\|_2}$ (2) => the square of the sum is not, on average, much smaller than the sum of squares.

=> There isn't too much cancellation in $\sum_I (f,h_I)h_I$.

To paraphrase: appealing to Bessel hides some of the magic going on here.

Emphasized the non-obviousness of $\|f\|_2 = \|S_D f\|_2$ because it is the starting point of Littlewood–Paley theory, whose goal is to extend it to settings where it's patently not obvious (L^p spaces, weighted spaces).

We will show:

Thm.: For all $1<p<\infty$, L^p-norms are equivalent to the square function:

$$\|S_D f\|_p \simeq \|f\|_p, \quad \forall p\in(0,1),\ f\in L^p(\mathbb{R})$$

i.e. there exist dimensional constants c_p, C_p such that

$$\boxed{c_p \|f\|_p \le \|S_D f\|_p \le C_p \|f\|_p,} \quad \forall f\in L^p(\mathbb{R}).$$

This actually gives us for free that $\|T_\sigma f\|_p \lesssim \|f\|_p$:

$$\|T_\sigma f\|_p \simeq \|S_D T_\sigma f\|_p \le \|S_D f\|_p \lesssim \|f\|_p$$

$$S_D^2 T_\sigma f(x) = \sum_{I\in D} \underbrace{\sigma_I^2}_{\le 1} (f,h_I)^2 \frac{\mathbb{1}_I(x)}{|I|} \le S_D^2 f(x).$$

Some necessary new notions first: (dyadic) maximal function, CZ decomposition, sharp function, good-λ inequalities...

Some remarks about the proof of $\|S_\mathcal{D} f\|_p \simeq \|f\|_p$, $1<p<\infty$:

→ It suffices to prove the upper bound $\|S_\mathcal{D} f\|_p \lesssim \|f\|_p$, $1<p<\infty$.
The lower bound then follows by duality:

$$\begin{aligned}
\|f\|_p &= \sup_{\substack{g\in L^{p'}\\ \|g\|_{p'}\le 1}} |(f,g)| = \sup_{\|g\|_{p'}\le 1} \left|\sum_I (f,h_I)(g,h_I)\right| \\
&\le \sup_{\|g\|_{p'}\le 1} \sum_I |(f,h_I)||(g,h_I)| = \sup_{\|g\|_{p'}\le 1} \int \left(\sum_I |(f,h_I)||(g,h_I)| \frac{\mathbb{1}_I}{|I|} dx\right) \\
&\le \sup_{\|g\|_{p'}\le 1} \int \left(\sum_I |(f,h_I)|^2 \frac{\mathbb{1}_I}{|I|}\right)^{1/2} \left(\sum_I |(g,h_I)|^2 \frac{\mathbb{1}_I}{|I|}\right)^{1/2} dx \\
&= \sup_{\|g\|_{p'}\le 1} \int (S_\mathcal{D} f)(S_\mathcal{D} g)\, dx \\
&\le \sup_{\|g\|_{p'}\le 1} \left(\|S_\mathcal{D} f\|_p \|S_\mathcal{D} g\|_{p'}\right) \lesssim \|S_\mathcal{D} f\|_p \sup_{\|g\|_{p'}\le 1} \|g\|_{p'} \le \|S_\mathcal{D} f\|_p.
\end{aligned}$$

→ The proof for the upper bound has 2 parts:

I. $1<p<2$: Interpolation with a weak 1,1 bound (proved via CZ decomposition)

II. $p>2$: Sharp function (good-λ inequalities).

Remark: One would maybe expect to obtain the $p>2$ result for free, using duality & adjoints. However, $S_\mathcal{D}$ is not linear, so that doesn't work here; the case $p>2$ has to be proved separately by hand.

For example: There is a direct proof for $\|T_\sigma f\|_p \lesssim \|f\|_p$ that doesn't appeal to the square function:

- $p=2$: $\|T_\sigma f\|_2^2 = \sum_I \sigma_I^2 (f,h_I)^2 \le \|f\|_2^2$.
- $1<p<2$: Interpolation with a weak $(1,1)$ bound (via CZ decomp.)
- $p>2$: Follows for free because $T_\sigma^* = T_\sigma$:

$$(T_\sigma f, g) = \sum_I \sigma_I (f,h_I)(g,h_I) = \left(f, \sum_I \sigma_I (g,h_I) h_I\right) = (f, T_\sigma g)$$

Let $p>2$. Then it's Hölder conjugate $p' \in (1,2)$ and $T_\sigma: L^{p'} \to L^{p'}$ is bounded $\Rightarrow \|T_\sigma: L^p \to L^p\| = \|T_\sigma^*: L^{p'} \to L^{p'}\| \lesssim \|f\|$.

→ The main point for proving part I. will be proving $S_\mathcal{D}: L^1 \to L^{1,\infty}$. After that, we appeal to Marcienkiewicz Interpolation Thm. (Grafakos, Section 1.3.a.) for quasilinear operators ($S_\mathcal{D}$)

$$\boxed{\left.\begin{array}{l} 0<p_0<p_1\le\infty \\ T \text{ quasilinear op. s.t.:} \\ \quad T: L^{p_0}(X) \to L^{p_0,\infty}(Y) \text{ b'dd} \\ \quad T: L^{p_1}(X) \to L^{p_1,\infty}(Y) \text{ b'dd} \end{array}\right\} \Rightarrow \begin{array}{l} T: L^p(X) \to L^p(Y) \\ \text{b'dd, } \forall\, p_0<p<p_1 \end{array}}$$

→ Applied to $S_\mathcal{D}$, knowing L^2-boundedness:

$$\left.\begin{array}{l} S_\mathcal{D}: L^1 \to L^{1,\infty} \\ S_\mathcal{D}: L^2 \to L^2 \subset L^{2,\infty} \end{array}\right\} \Rightarrow \begin{array}{l} S_\mathcal{D}: L^p \to L^p \\ 1<p<2 \end{array}$$